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F-312

M.A./M.Sc. (First Semester)
EXAMINATION, Dec. - Jan., 2021-22
MATHEMATICS
Paper Fourth
(Advanced Complex Analysis - I)

Time : Three Hours] ***[Maximum Marks:80***

[Minimum Pass marks:16

Note : Attempt all sections as directed.

Section - A

(Objective/Multiple Choice Questions)

(1 mark each)

Note: Attempt all questions.

Choose the correct answer:

1. The path of the difinite integral $\int_a^b f(z)dz$ is :

   (A) The line segment joining the points z = a and z = b.

   (B) Any curve joining the points z = a and z = b.

   (C) Any circle such that the points z = a and z = b lie on it.

   (D) None of these.

P.T.O.

2. If f (z) is analytic in a simply connected domain D and C is any closed continuous rectifible curve in D, then $\int_c f(z)dz$ is equal to -

   (A) 0 (B) 1

   (C) C (D) D

3. For the function $f(z)=\tan\frac{1}{z},\ z=0$ is:

   (A) Isolated essential singularities

   (B) Removable singularity

   (C) Non - isolated essential singularity

   (D) None of these

4. Poles of an analytic function are:

   (A) Isolated

   (B) Non - isolated

   (C) Removable

   (D) None of these

5. If $f(z)$ is analytic in a domain $|z|<1$ and satisfies the conditions $f(z)\leq 1,\ f(0)=0$ then:

   (A) $|f(z)|\geq z, |f'(0)|\leq 1$

   (B) $|f(z)|\geq z, |f'(0)|\geq 1$

   (C) $|f(z)|\leq z, |f'(0)|\leq 1$

   (D) All of the above

6. One of the roots of the equation $Z^4 + Z^3 + 1 = 0$ lies in the:

(A) First Quadrant

(B) Second Quadrant

(C) First & Second Quadrant

(D) None of these

7. The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-1}}{(2n-1)!}$ when $|z| < \infty$ represents.

(A) Sin z

(B) Cos z

(C) Log (1-z)

(D) None of these

8. At z = 1, the function $f(z) = \frac{z}{z^2 - 1}$ has a pole of order:

(A) One

(B) Two

(C) No pole exists

(D) None of these

9. If f (z) has an isolated singularity at $z = \infty$ then the residue at $z = \infty$ is

(A) $\frac{1}{2\pi i}\int_c f(z)dz$ (B) $\frac{-1}{2\pi i}\int_c f(z)dz$

(C) $\frac{\pm 1}{2\pi i}\int_c f(z)dz$ (D) None of these

Here C is any closed contour which encloses all the finite singularities of f (z) & integral is taken in positive direction.

10. The residue of $f(z) = \frac{e^z}{z^2(z^2 + a)}$ at z = o is-

(A) 0 (B) ie^{3i}

(C) $\frac{1}{9}$ (D) $\frac{3}{13}$

11. The number of poles of $f(z) = \frac{1}{z(z^2+3)(z^2+2)^3}$ inside the circle $|z| = 1$ are -

(A) 9 (B) 5

(C) 2 (D) 1

12. The residue of $\frac{1}{(Z^2+1)^3}$ $at\, z = i$ is given by -

(A) $\frac{3}{8i}$

(B) $\frac{3i}{8}$

(C) $\frac{3}{16i}$

(D) $\frac{3i}{16}$

13. A Transformation of the type $\omega = \alpha z + \beta$, where α and β are complex constant, is known as a:

(A) Translation

(B) Magnification

(C) Linear transformation

(D) Bilinear transformation

14. Under the transformation $\omega = \frac{1}{z}$, the image of the line $y = 1/4$ in z - plane is:

(A) Circle $u^2 + v^2 = 4$

(B) Straight line

(C) Circle $u^2 + \upsilon^2 + 4\upsilon = 0$

(D) None of them

15. If $\omega = f(z)$ represents a conformal mapping of a domain D, then f(z) is:

(A) Analytic in D

(B) Not necessarily analytic in D

(C) Not analytic in D

(D) None of these

16. The fixed points of the bilinear transformation $\omega = \frac{z}{z-2}$ are:

(A) 0,0

(B) 0,3

(C) 0,2

(D) None of these

17. Let $\{fn\}$ be a sequence in H (G) and $f \in c(G,C)$ such that $fn \to f$. Then f is analytic and $f_n^{(k)} n \to f^{(k)}$ for each integer-

(A) $k \le 1$ (B) $k = 0$

(C) $k \ge 1$ (D) None of these

18. If f is analytic in a domain D and is not constant then $\omega = f(z)$ maps open sets of D onto -

(A) Open sets in ω - plane

(B) Closed set in ω - plane

(C) (A) and (B) both

(D) None of these

19. The space H(G) of analytic functions of G is a:

(A) Metric space

(B) Complete metric space

(C) Not necessarily complete

(D) None of these

20. Let F $\subset$ C (G, Ω) (the set of all continuous functions from G to Ω). If each sequence in F has a subsequence which converges to a function f in C (G, Ω). Then F is called -

(A) Totally bounded

(B) Compact

(C) Normal

(D) Locally bounded

Section - B

(Very Short Answer Type Questions)

(2 marks each)

Note : Attempt all questions.

1. Evaluate $\oint_c \frac{e^z}{(z-1)(Z-4)}dz$ where C is the circle $|z| = 2$ by using Cauchy's integral formula.
2. Write the statement of Morera's Theorem.
3. Write the statement of Minimum Modulus Principle.
4. Find the residue of $\frac{1}{(z^2+1)^3}$ at $z = i$
5. Define meromorphic function.
6. Consider the transformation $\omega = T(z) = \frac{z+1}{z+3}$ $find\ T^{-1}(\omega)$.
7. Write the sufficient condition for $\omega = f(z)$ to represent a conformal mapping.
8. State the Riemann mapping theorem.

Section - C

(Short Answer Type Questions)

(3 marks each)

Note: Attempt all questions.

1. Prove that $\cos h\left(z+\frac{1}{z}\right) = a_0 + \sum_{n=1}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

   where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos h(2\cos\theta)\cos n\theta\, d\theta$

2. Prove that the value of the integral of $\frac{1}{Z}$ along a semi - circular arc $|z| = a$ from - a to a is $-\pi i$ or πi according as the arc lies above or below the real axis.
3. State the argument principle.
4. By the method of contour integration, show that $\int_0^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2}$
5. Consider the transformation $\omega = 3z$ and determine the region D' of the co-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ in the z - plane is mapped under this transformation.
6. Find the bilinear transformation which maps 0,1 and ∞ into 1, i and - 1respectively.
7. Show that if a set F$\subset$C (G, Ω) is normal then $\overline{F}$ is normal.
8. Show that (C (G,Ω),ρ) is a metric space.

Section - D

(Long Answer Type Questions)(5 mark each)

Note: Attempt all questions.

1. State and prove Cauchy's integral formula for higher order derivative.

OR

If f(z) is analytic within and on a closed contour C except at a finite number of poles and has no zero on C, then prove that, $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz = N - P,$

Where N is the number of zeros and P the number of poles inside C and a pole or zero of order m being counted m times.

2. By contour integration, show that:

$$\int_0^\infty \frac{\sin x}{x(x^2+a^2)} dx = \frac{\pi}{2a^2}(1-e^{-a}), \ (a>0)$$

OR

State and prove Cauchy's Residue theorem.

3. Show that in the transformation $(\omega+1)^2 = \frac{4}{z}$, the unit circle in the ω - plane corrsponds to a parabola in z - plane and inside of the circle to the outside of the parabola.

OR

Find the bilinear transformations which maps the half plane $I(z) \geq 0$ onto the unit circular disc $|\omega| \leq 1.$

4. State and prove Montel's theorem.

OR

State and prove open mapping theorem.